

ZEROth POISSON HOMOLOGY OF SYMMETRIC POWERS OF ISOLATED QUASIHOMOGENEOUS SURFACE SINGULARITIES

PAVEL ETINGOF AND TRAVIS SCHEDLER

ABSTRACT. Let $X \subset \mathbb{C}^3$ be a surface with an isolated singularity at the origin, given by the equation $Q(x, y, z) = 0$, where Q is a weighted-homogeneous polynomial. In particular, this includes the Kleinian surfaces $X = \mathbb{C}^2/G$ for $G < \mathrm{SL}_2(\mathbb{C})$ finite. Let $Y := S^n X$ be the n -th symmetric power of X . We compute the zeroth Poisson homology $\mathrm{HP}_0(Y)$, as a graded vector space with respect to the weight grading. In the Kleinian case, this confirms a conjecture of Alev, that $\mathrm{HP}_0(\mathbb{C}^{2n}/(G^n \rtimes S_n)) \cong \mathrm{HH}_0(\mathrm{Weyl}_{2n}^{G^n \rtimes S_n})$, where Weyl_{2n} is the Weyl algebra on $2n$ generators. That is, the Brylinski spectral sequence degenerates in this case. In the elliptic case, this yields the zeroth Hochschild homology of symmetric powers of the elliptic algebras with three generators modulo their center, A_γ , for all but countably many parameters γ in the elliptic curve.

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1. INTRODUCTION

1.1. Main result. Let a, b, c be positive integers, and equip $\mathbb{C}[x, y, z]$ with a weight grading in which $|x| = a, |y| = b$, and $|z| = c$. In this paper, we are interested in surfaces $X \subset \mathbb{C}^3$ with an isolated singularity at the origin, cut out by a polynomial $Q(x, y, z) = 0$, which is weighted-homogeneous of degree d . Such surfaces were first studied systematically by Saito [Sai87]. For convenience, we also assume that $a \leq b \leq c$.

The surface X is equipped with a standard Poisson bracket, given by the bivector

$$(1.1.1) \quad \pi := \left(\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} \right) \lrcorner (dQ),$$

where \lrcorner is the natural contraction operation, which in this case produces a bivector from a trivector and a one-form. The above bivector is, moreover, weight-homogeneous of degree $\kappa := d - (a + b + c)$, and is a Poisson bivector (i.e., $\{\pi, \pi\} = 0$, where $\{\cdot, \cdot\}$ is the Schouten-Nijenhuis bracket). Hence it produces a Poisson bracket of degree κ .

In particular, when $\kappa < 0$, X has a Kleinian singularity, i.e., $X \cong \mathbb{C}^2/G$ where $G < \mathrm{SL}_2(\mathbb{C})$ is a finite subgroup. These finite subgroups have a well-known classification, and up to equivalence, we have:

$$(1.1.2) \quad A_{m-1} : G = \mathbb{Z}/m, a = 2, b = c = m, Q = x^m + y^2 + z^2,$$

$$(1.1.3) \quad D_{m+2} : G = \widetilde{D_{2m}}, a = 2, b = m, c = m + 1, Q = x^{m+1} + xy^2 + z^2,$$

$$(1.1.4) \quad E_6 : G = \widetilde{A_4}, a = 3, b = 4, c = 6, Q = x^4 + y^3 + z^2,$$

$$(1.1.5) \quad E_7 : G = \widetilde{S_4}, a = 4, b = 6, c = 9, Q = x^3y + y^3 + z^2,$$

$$(1.1.6) \quad E_8 : G = \widetilde{A_5}, a = 6, b = 10, c = 15, Q = x^5 + y^3 + z^2.$$

Here we set the degree κ of the bracket to -2 in the A case and -1 for the D and E cases.

The case $\kappa = 0$ (i.e., $d = a + b + c$) is called the *elliptic* case, and, up to equivalence, the surface has one of the following forms, for some $\lambda \in \mathbb{C}^\times$:

$$(1.1.7) \quad \widetilde{E}_6 : a = b = c = 1, Q = x^3 + y^3 + z^3 + \lambda xyz,$$

$$(1.1.8) \quad \widetilde{E}_7 : a = b = 1, c = 2, Q = x^4 + y^4 + z^2 + \lambda xyz,$$

$$(1.1.9) \quad \widetilde{E}_8 : a = 1, b = 2, c = 3, Q = x^6 + y^3 + z^2 + \lambda xyz.$$

Let $X^{(n)} := S^n X$ be the n -th symmetric power of X , which is a singular affine Poisson variety. In this paper, we compute explicitly the zeroth Poisson homology of $X^{(n)}$, as a graded vector space using the weight grading. To describe this, recall the Jacobi ring of X ,

$$(1.1.10) \quad J_Q := \mathbb{C}[x, y, z]/(Q_x, Q_y, Q_z),$$

where Q_x, Q_y , and Q_z are the partial derivatives of Q with respect to x, y , and z , respectively. Then, J_Q is finite-dimensional, and its dimension is called the *Milnor number*, and denoted by μ_Q .

For any graded vector space V with finite-dimensional graded components, let $h(V; t)$ denote its Hilbert series.

Recall that, for any Poisson algebra A , its zeroth Poisson homology is defined as

$$(1.1.11) \quad \mathrm{HP}_0(A) := A/\{A, A\}.$$

In [AL98], Alev and Lambre showed that

$$(1.1.12) \quad \mathrm{HP}_0(\mathcal{O}_X) \cong J_Q,$$

as weight-graded vector spaces. Here \mathcal{O}_X denotes the global functions on X (which is affine).

It will be convenient to combine the linear duals of the homology groups $\mathrm{HP}_0(\mathcal{O}_{X^{(n)}})$, for $n \geq 0$, into one bigraded algebra $\bigoplus_{n \geq 0} \mathrm{HP}_0(\mathcal{O}_{X^{(n)}})^*$, with the multiplication given by the symmetrization maps $\mathcal{O}_{X^{(i)}}^* \otimes \mathcal{O}_{X^{(j)}}^* \rightarrow \mathcal{O}_{X^{(i+j)}}^*$. The main result of this paper is

Theorem 1.1.13. $\bigoplus_{n \geq 0} \mathrm{HP}_0(\mathcal{O}_{X^{(n)}})^*$ is isomorphic, as a bigraded algebra, to a free commutative algebra generated by a bigraded vector space L with Hilbert series

$$(1.1.14) \quad h(L; t^{-1}, s) = \frac{h(J_Q; t)s}{1 - t^d s}.$$

Here, the exponent of t is the weight, and the exponent of s is the corresponding symmetric power of X . Note that there is a t^{-1} since, by convention, the weights are negated when we take the dual.

We may thus write the Hilbert series of $\bigoplus_{n \geq 0} \mathrm{HP}_0(\mathcal{O}_{X^{(n)}})$ itself by the following formula. Write $h(J_Q; t) = t^{n_1} + \dots + t^{n_r}$ (in the Kleinian case, the numbers $m_i := n_i + 1$ (in types D, E) or $m_i := \frac{n_i}{2} + 1$ (in type A) are the Coxeter exponents associated to the corresponding finite Weyl group of type ADE, which has Coxeter number $h = d$ (types D, E) or $h = \frac{d}{2}$ (type A)).

$$(1.1.15) \quad h\left(\bigoplus_{n \geq 0} \mathrm{HP}_0(\mathcal{O}_{X^{(n)}}); t, s\right) = \prod_{i=1}^r \prod_{j \geq 0} \frac{1}{1 - t^{n_i + j d} s^{j+1}}.$$

1.2. Hochschild homology of deformations and Alev's conjecture. In the Kleinian case, i.e., (1.1.2)–(1.1.6), $X \cong \mathbb{C}^2/G$ for the listed finite subgroup $G < \mathrm{SL}_2(\mathbb{C})$. Using this, one has a canonical quantization of $\mathcal{O}_{X^{(n)}}$, namely $\mathrm{Weyl}_{2n}^{G^n \rtimes S_n} = \mathrm{Sym}^n \mathrm{Weyl}_2^G$, where the Weyl algebras are defined as $\mathrm{Weyl}_2 = \mathbb{C}\langle x, y \rangle / ([x, y] - 1)$ and $\mathrm{Weyl}_{2n} = \mathrm{Weyl}_2^{\otimes n}$, and $G^n \rtimes S_n$ is the semidirect product where S_n acts by permuting components (i.e., the wreath product of G with S_n), viewed as a subgroup of $\mathrm{Sp}_{2n}(\mathbb{C})$.

Here, by “quantization” of a graded Poisson algebra B with bracket of degree $-f$ for some $f > 0$, we mean a filtered associative algebra $A = \bigcup_m A_{\leq m}$ such that $\mathrm{gr}(A) = B$, and such that, for every $a \in A_{\leq m}, b \in A_{\leq n}$, we have $ab - ba \in A_{\leq (m+n-f)}$ and the image of $ab - ba$ in $\mathrm{gr}_{m+n-f}(A)$ is $\{\mathrm{gr}_m(a), \mathrm{gr}_n(b)\}$. (We could alternatively define this as a deformation quantization satisfying a homogeneity property.)

In this situation, there is a standard Brylinski spectral sequence from $\mathrm{HP}_0(B)$ to $\mathrm{HH}_0(A) := A/[A, A]$. Moreover, we may equip $\mathrm{HH}_0(A)$ with the weight filtration from A , and it is easy to see that this spectral sequence preserves the grading, in the sense that each page consists of homogeneous differentials. Thus, the spectral sequence converges to $\mathrm{gr} \mathrm{HH}_0(A)$.

One may ask whether the Brylinski spectral sequence degenerates. As a consequence of our main theorem, we may deduce the

Theorem 1.2.1. In the case that $B = \mathcal{O}_{X^{(n)}}$ for $X = \mathbb{C}^2/G$ a Kleinian singularity listed in (1.1.2)–(1.1.6), and $A = \mathrm{Weyl}_{2n}^{G^n \rtimes S_n}$, the Brylinski spectral sequence $\mathrm{HP}_0(B) \Rightarrow \mathrm{gr} \mathrm{HH}_0(A)$ degenerates, yielding an isomorphism of graded vector spaces, $\mathrm{HP}_0(B) \cong \mathrm{gr} \mathrm{HH}_0(A)$.

This confirms a conjecture of J. Alev [But08, Remark 40]. In the case where $G = \mathbb{Z}/2 = \{\pm \mathrm{Id}\} \subset \mathrm{Sp}_2(\mathbb{C})$, this was proved in the case $n = 2$ in [AF], and for $n = 3$ in [But08], where also some preliminary results and conjectures are given towards general n (again for $G = \mathbb{Z}/2$).

We remark that Alev also posed a similar conjecture, which replaces $G^n \rtimes S_n$ as above with finite Weyl groups $W < \mathrm{GL}_n(\mathbb{C}) \hookrightarrow \mathrm{Sp}_{2n}(\mathbb{C})$, where the embedding is given by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}$. The case $(\mathbb{Z}/2)^n \rtimes S_n$ then identifies with the Weyl groups of type B_n (and

also with type C_n). We do not know whether this conjecture holds for general Weyl groups of types other than B (or C), although it was verified in types $D_2 = A_1 \times A_1$ and $D_3 = A_3$ in [But08].

To deduce Theorem 1.2.1 from Theorem 1.1.13, one uses [AFLS00], which gives a general formula for the dimension of $\mathrm{HH}_0(\mathrm{Weyl}_{2n}^G)$ for arbitrary n and $G < \mathrm{Sp}_{2n}(\mathbb{C})$. However, we will use only the $n = 1$ case and a general result from [EO06, §3] to make this more transparent.

In the non-Kleinian cases, the Poisson bracket does not have negative degree, so the above does not apply. However, following [EG07] (e.g., Theorems 3.4.4 and 3.4.5), one may always produce a deformation quantization of \mathcal{O}_X , i.e., a $\mathbb{C}[[\hbar]]$ -algebra A_\hbar which is isomorphic to $\mathcal{O}_X[[\hbar]]$ as a $\mathbb{C}[[\hbar]]$ -module, such that $A_\hbar/(\hbar) \cong \mathcal{O}_X$ and such that $[a, b] = \hbar\{a, b\} + O(\hbar^2)$ for all $a, b \in \mathcal{O}_X \subset \mathcal{O}_X[[\hbar]] \cong A_\hbar$. For such a deformation quantization, we may similarly deduce the

Theorem 1.2.2. The Brylinski spectral sequence $\mathrm{HP}_0(\mathcal{O}_{X^{(n)}})((\hbar)) \Rightarrow \mathrm{gr} \mathrm{HH}_0(\mathrm{Sym}^n A_\hbar[\hbar^{-1}])$ degenerates.

This generalizes the theorem above.

In the case that the Poisson bracket has degree zero, there exist not merely formal but actual, *homogeneous* quantizations of A , the Artin-Tate-Odesski-Sklyanin-type algebras A_γ modulo their center (e.g., [ATdB90, Ste97]; see also [EG07, §3.5]). Here the parameter \hbar is replaced by a point γ on an elliptic curve. For such algebras, we deduce

Corollary 1.2.3. For all but countably many parameters γ , we have a noncanonical isomorphism of weight-graded vector spaces, $\mathrm{HH}_0(\mathrm{Sym}^n A_\gamma) \cong \mathrm{HP}_0(\mathcal{O}_{X^{(n)}})$, and moreover,

$$(1.2.4) \quad \bigoplus_n \mathrm{HH}_0(\mathrm{Sym}^n A_\gamma)^* \cong \mathrm{Sym}(\mathrm{HH}_0(A_\gamma)^*[t]),$$

as bigraded algebras (noncanonically), where $\mathrm{HH}_0(\mathrm{Sym}^n A_\gamma)^*$ has degree n , and t has degree 1 and weight $-d$.¹

Note that, when γ is a point of finite order of the elliptic curve, then $\mathrm{HH}_0(A_\gamma)$ is infinite-dimensional, and the isomorphism $\mathrm{HH}_0(A_\gamma) \cong \mathrm{HP}_0(\mathcal{O}_X)$ fails. We expect that these are exactly the countably many γ mentioned in the corollary.

Moreover, we deduce the same result for the filtered deformations of these elliptic algebras as in [VdB01, EG07]:

Corollary 1.2.5. For all but countably many γ , if A'_γ is an associative algebra with an ascending filtration such that $\mathrm{gr} A'_\gamma = A_\gamma$, then $\mathrm{gr} \mathrm{HH}_0(\mathrm{Sym}^n A'_\gamma) \cong \mathrm{HH}_0(\mathrm{Sym}^n A_\gamma)$ as graded vector spaces.

Remark 1.2.6. The isomorphisms above don't have anything to do with the specific quantizations. Generally, the n -th symmetric power of any formal or generic filtered (or graded) quantization A' of \mathcal{O}_X has $\mathrm{HH}_0(\mathrm{Sym}^n A') \cong \mathrm{HP}_0(\mathrm{Sym}^n \mathcal{O}_X) \otimes K$, as filtered (or graded) vector spaces, where K is the base field for the deformation (i.e., $K = \mathbb{C}((\hbar))$ for a formal deformation, and $K = \mathbb{C}$ for a generic enough point of an actual deformation). Similarly, for any formal or generic filtered/graded Poisson deformation B' of \mathcal{O}_X , we have

¹In general, there is a canonical map from the RHS to the LHS as degree-graded algebras, but it is *not* an isomorphism, nor a map of weight-graded algebras; see Remark 5.0.78.

$\mathrm{HP}_0(\mathrm{Sym}^n B') \cong \mathrm{HP}_0(\mathrm{Sym}^n \mathcal{O}_X) \otimes K$, as filtered/graded vector spaces. Note that the semi-universal formal deformations were classified in [EG07], and the actual elliptic deformations were deduced as well (these were studied earlier by a different method in [VdB01]).

1.3. General symmetric products and Poisson-invariant functionals. It is convenient to replace Poisson homology with invariant functionals, as follows. Let X be an affine Poisson variety. Let $\mathfrak{g}_X \subseteq \Gamma(X, TX)$ be the Lie algebra of Hamiltonian vector fields, which is the same as $\mathcal{O}_X/Z(\mathcal{O}_X)$, viewing \mathcal{O}_X as a Lie algebra, and denoting by $Z(\mathcal{O}_X)$ the Poisson center of \mathcal{O}_X . As is standard, invariants of a Lie algebra action of \mathfrak{g} on A are denoted by $A^{\mathfrak{g}} := \{a \in A : g(a) = 0, \forall g \in \mathfrak{g}\}$.

We now reformulate the problem of computing Poisson homology of the varieties $X^{(n)}$:

Proposition 1.3.1. For every affine Poisson variety X , there is a canonical graded algebra isomorphism

$$(1.3.2) \quad \bigoplus_{n \geq 0} \mathrm{HP}_0(X^{(n)})^* \cong \mathbb{C}[\mathcal{O}_X]^{\mathfrak{g}_X} = \bigoplus_{n \geq 0} (\mathrm{Sym}^n(\mathcal{O}_X)^*)^{\mathfrak{g}_X}.$$

Here, $\mathbb{C}[\mathcal{O}_X]$ denotes the polynomial functions on the (infinite-dimensional) vector space \mathcal{O}_X , which is equipped with the coadjoint action of \mathfrak{g}_X . The proof is easy and short:

Proof. We have $\mathrm{HP}_0(X^{(n)}) = (\mathrm{Sym}^n \mathcal{O}_X)/\{\mathrm{Sym}^n \mathcal{O}_X, \mathrm{Sym}^n \mathcal{O}_X\}$. We claim that there is an identification

$$(1.3.3) \quad \{\mathrm{Sym}^n \mathcal{O}_X, \mathrm{Sym}^n \mathcal{O}_X\} = \mathfrak{g}_X(\mathrm{Sym}^n \mathcal{O}_X).$$

Note that we have an obvious inclusion of vector spaces, $\mathcal{O}_X \subseteq \mathrm{Sym}^n \mathcal{O}_X$, given by $f \mapsto f \& 1 \& \cdots \& 1$. Using this, the inclusion \supseteq above follows from the equality

$$(1.3.4) \quad g(f_1 \& f_2 \& \cdots \& f_n) = \{g, f_1 \& \cdots \& f_n\}.$$

For the inclusion \subseteq , we use the fact that $\mathrm{Sym}^n \mathcal{O}_X$ is generated, as a commutative algebra, by the subspace $\mathcal{O}_X \subseteq \mathrm{Sym}^n \mathcal{O}_X$: this follows inductively on n by an easy argument. Then, we use the fact that, for any Poisson algebra A which is generated as a commutative algebra² by $V \subseteq A$, we have $\{A, A\} = \{V, A\}$, by the identity

$$(1.3.5) \quad \{ab, c\} = \{a, bc\} + \{b, ca\}. \quad \square$$

1.4. A \mathbb{C}^* -equivariant vector bundle on \mathbb{P}^1 . We return to the surface $X = Z(Q)$ from §1.1. In the Kleinian A_{m-1} -case, we will give a short and self-contained proof of Theorem 1.1.13 in §4 (which would deserve mention even if the proof below extended to this case).

The main step of the proof for all other cases is to write $\mathbb{C}[\mathcal{O}_X]^{\mathfrak{g}_X}$ as the algebra of global sections of a certain infinite-dimensional \mathbb{C}^* -equivariant vector bundle on \mathbb{P}^1 , whose definition and structure we explicitly describe in this section.

Henceforth, we assume that we are not in the Kleinian type A_{m-1} case. This has the following important consequence:

Lemma 1.4.1. Suppose that X is not of Kleinian type A_{m-1} . Then, all nonzero homogeneous Hamiltonian vector fields ξ_f have positive degree. In particular, $\langle x, y \rangle \cap \{\mathcal{O}_X, \mathcal{O}_X\} = 0$.

²In fact, $\{A, A\} = \{V, A\}$ assuming only that A is generated as a Poisson algebra by V , using (1.3.5) and the Jacobi identity.

Proof. In the non-Kleinian case, it is clear that ξ_f has positive degree for all noncentral f , since the Poisson bracket has nonnegative degree and \mathbb{C} is central. In the Kleinian case, looking at (1.1.2)–(1.1.6), only in the type A_{m-1} case is there a Hamiltonian vector field of nonpositive degree (in particular, ξ_x has zero degree, and in type A_1 , also ξ_y and ξ_z). Then, since $a \leq b \leq c$, for all homogeneous f, g with $\{f, g\} \neq 0$, we have $|\{f, g\}| \geq |\xi_x(y)| > b$, which implies the final statement. \square

Next, for any $m \geq 0$, let $(\mathcal{O}_X)_m$ denote the subspace of \mathcal{O}_X of weighted degree m . It is convenient to consider, rather than $\mathbb{C}[\mathcal{O}_X]$, the subalgebra

$$(1.4.2) \quad \mathcal{F}(X) := \text{Sym}\left(\bigoplus_{m \geq 0} ((\mathcal{O}_X)_m)^*\right)$$

The entire algebra $\mathbb{C}[\mathcal{O}_X]$ is the completion of $\mathcal{F}(X)$ by the weight grading. In other words, $\mathcal{F}(X)$ is the algebra of continuous polynomial functions on the completion $\mathcal{O}_{\widehat{X}} = \mathbb{C}[[x, y, z]]/(Q)$ with respect to the weight grading. We may view $\mathcal{O}_{\widehat{X}}$ as a pro-scheme (with limit taken over finite-dimensional affine spaces), and in this sense, $\mathcal{F}(X) = \mathbb{C}[\mathcal{O}_{\widehat{X}}]$. Note that $\mathbb{C}[\mathcal{O}_X]^{\text{gx}}$ is also a completion of $\mathcal{F}(X)^{\text{gx}}$. In our case, in fact, $(\text{Sym}^n(\mathcal{O}_X)^*)^{\text{gx}}$ will turn out to be finite-dimensional for each n , and hence $\mathbb{C}[\mathcal{O}_X]^{\text{gx}} = \mathbb{C}[\mathcal{O}_{\widehat{X}}]^{\text{gx}}$.

Let $V := \langle x, y \rangle \subset \mathcal{O}_{\widehat{X}}$. Fix a graded complement $\mathcal{O}_{\widehat{X}}^0$ to V containing all Poisson brackets. Thus, $\mathcal{O}_{\widehat{X}} = V \times \mathcal{O}_{\widehat{X}}^0$. Note that functionals in $\mathcal{F}(X)$ are the same as regular functions on $(V \setminus \{0\}) \times \mathcal{O}_{\widehat{X}}^0$.

Lemma 1.4.3. The invariants $\mathcal{F}(X)^{\text{gx}}$ can be noncanonically identified with regular functions on the total space of the pro-vector bundle Y' on $V \setminus \{0\}$ with fiber over (α, β) given by

$$(1.4.4) \quad Y'_{(\alpha, \beta)} = \mathcal{O}_{\widehat{X}}^0 / \{\alpha x + \beta y, \mathcal{O}_{\widehat{X}}\}.$$

The lemma will be proved in Section 3. Let us explain why (1.4.4) indeed defines a pro-vector bundle. Note that Y' is a pro-coherent sheaf which is pulled back from \mathbb{P}^1 . Next, viewing $\mathcal{O}_{\widehat{X}}$ as a constant pro-vector bundle, we may view Y' as the cokernel of the pro-coherent sheaf map,

$$(1.4.5) \quad \mathcal{O}_{\widehat{X}} \otimes \mathcal{O}(-1) \rightarrow \mathcal{O}_{\widehat{X}}^0, \quad f \otimes (\alpha x + \beta y) \mapsto \{f, \alpha x + \beta y\}.$$

This map descends to

$$(1.4.6) \quad \mathcal{O}_{\widehat{X}} / \mathbb{C}[[\alpha x + \beta y]] \otimes \mathcal{O}(-1) \rightarrow \mathcal{O}_{\widehat{X}}^0,$$

where $\mathcal{O}_{\widehat{X}} / \mathbb{C}[[\alpha x + \beta y]]$ is the quotient of $\mathcal{O}_{\widehat{X}}$ by the sub-pro-vector bundle $\mathbb{C}[[\alpha x + \beta y]] \cong \prod_{i \geq 0} \mathcal{O}(-1)^{\otimes i}$.

We claim that (1.4.6) is injective on fibers. This follows by computing that $\mathbb{C}[[\alpha x + \beta y]]$ is the kernel of $\{\alpha x + \beta y, -\}$, see Lemma 2.0.19. Hence, this is a pro-vector bundle map, and the cokernel, Y' , is indeed a pro-vector bundle.

Next, note that Y' is equipped with a \mathbb{C}^* -equivariant structure with respect to the \mathbb{C}^* -action on V , given by, for $w \in \mathbb{C}^*$,

$$(1.4.7) \quad x \mapsto w^a x, \quad y \mapsto w^b y.$$

The action on coordinate functions then has the form

$$(1.4.8) \quad \alpha \mapsto w^{-a} \alpha, \quad \beta \mapsto w^{-b} \beta.$$

Furthermore, Y' is pulled back from a pro-vector bundle Y on \mathbb{P}^1 . We may thus regard $\mathcal{F}(X)^{\mathfrak{g}_X}$ as the regular functions on the total space of the pro-vector bundle

$$(1.4.9) \quad E := Y \oplus \mathcal{O}(-1)$$

on \mathbb{P}^1 . These pro-bundles are also \mathbb{C}^* -equivariant.

Note that representations W of \mathbb{C}^* may also be viewed as graded vector spaces, with action of w in degree m by multiplication by w^m . Thus, we will use the notation $h(W; t)$ for the character of W viewed as a representation of \mathbb{C}^* , i.e., the Hilbert series of W where W is viewed as a graded vector space (rather than a vector space with \mathbb{C}^* -action).

Next, we describe the structure of Y , which will imply the main theorem. First, recall the following basic facts about \mathbb{C}^* -equivariant vector bundles on \mathbb{P}^1 :

Definition 1.4.10. Let $\mathcal{O}(n)_m$ denote $\mathcal{O}(n)$ with the \mathbb{C}^* -equivariant structure given by the action of $w \in \mathbb{C}^*$ on the fiber over $(1, 0)$ as multiplication by w^m .

In particular, the tautological line bundle $\mathcal{O}(-1)$ (which appeared in (1.4.9)) is the equivariant bundle $\mathcal{O}(-1)_a$. We will need the following well-known result, whose proof is easy and omitted:

Theorem 1.4.11. Let \mathbb{P}^1 be equipped with the above \mathbb{C}^* -action.

- (i) Up to isomorphism, any \mathbb{C}^* -equivariant vector bundle on \mathbb{P}^1 has a unique decomposition as a sum of line bundles of the form $\mathcal{O}(n)_m$.
- (ii) For $n \geq 0$,

$$(1.4.12) \quad h(\Gamma(\mathbb{P}^1, \mathcal{O}(n)_m); t) = t^m(1 + t^{a-b} + \dots + t^{n(a-b)}).$$

Remark 1.4.13. In fact, we will work also with pro- \mathbb{C}^* -equivariant vector bundles, but only those for which the weight- m subspaces of the fibers at $(0, 1)$ and $(1, 0)$ are finite-dimensional for all $m \in \mathbb{Z}$. In this case, the above theorem still applies, except that now the pro-bundles will be a direct product of possibly infinitely many $\mathcal{O}(n)_m$ (but only finitely many for each value of m). In particular, the Hilbert series of global sections makes sense.

We may therefore make the following definition:

Definition 1.4.14. For any \mathbb{C}^* -equivariant vector bundle U on \mathbb{P}^1 of the form $U \cong \bigoplus_i \mathcal{O}(p_i)_{q_i}$, write

$$(1.4.15) \quad \chi_{\mathbb{C}^*}(U) = \sum_i s^{p_i} t^{q_i}.$$

We extend this notation in the obvious way to pro- \mathbb{C}^* -equivariant vector bundles whose fibers over $(0, 1)$ and $(1, 0)$ have finite-dimensional weight- m subspaces for all $m \in \mathbb{Z}$. Now, we may state the main technical result of the paper, which implies Theorem 1.1.13. It will be convenient to use the pro-bundle

$$(1.4.16) \quad \tilde{Y} := Y \oplus \mathcal{O}(0)_a \oplus \mathcal{O}(0)_b.$$

Theorem 1.4.17. We have

$$(1.4.18) \quad \chi_{\mathbb{C}^*}(\tilde{Y}) = \frac{(1 - t^{d-a})(1 - t^{d-b})(1 - t^{d-c})}{(1 - t^a)(1 - t^b)(1 - t^c)(1 - t^{d-a-s})}$$

The next section is devoted to the proof of this theorem.

2. PROOF OF THEOREM 1.4.17

The following lemma will be a cornerstone of the proof:

Lemma 2.0.19. The kernel of $\{\alpha x + \beta y, -\} : \mathcal{O}_{\hat{X}} \rightarrow \mathcal{O}_{\hat{X}}$ is $\mathbb{C}[[\alpha x + \beta y]] \subset \mathcal{O}_{\hat{X}}$.

(Note that the inclusion $\mathbb{C}[[\alpha x + \beta y]] \subset \mathcal{O}_{\hat{X}}$ makes sense since, e.g., $Q \notin \mathbb{C}[[x, y]]$).

Proof. We first claim that it is sufficient to consider the case where either $\alpha = 0$ or $\beta = 0$. First, if $a = b$ (i.e., $|x| = |y|$), then we may change bases to replace $\alpha x + \beta y$ with x . If $a < b$, then, letting Z_f denote the Poisson centralizer of f , we have $\text{gr } Z_{\alpha x + \beta y} \subseteq Z_x$ when $\alpha \neq 0$. Since $\mathbb{C}[[\alpha x + \beta y]] \subseteq Z_{\alpha x + \beta y}$, when $\alpha \neq 0$ it is sufficient to assume that $\beta = 0$.

Suppose $\beta = 0$. Let $f \in \mathcal{O}_{\hat{X}}$ be such that $\{x, f\} = 0$. We need to show that f is a power series in x . Clearly, one may assume without loss of generality that f is a polynomial. Because X is generically symplectic, f must be algebraically dependent on x . But it is easy to show that $\mathbb{C}[x]$ is algebraically closed in $\mathbb{C}[x, y, z]/(Q)$ (e.g., any homogeneous element in the algebraic closure of $\mathbb{C}[x]$ would have to be a rational power of x , and only nonnegative integer powers of x occur in $\mathbb{C}[x, y, z]/(Q)$), so $f \in \mathbb{C}[x]$. The case where $\alpha = 0$ is similar. \square

Now, in view of Theorem 1.4.11, to prove Theorem 1.4.17 it suffices to compute the character (Hilbert series) of the vector spaces

$$(2.0.20) \quad V_q := \Gamma(\mathbb{P}^1, \tilde{Y} \otimes \mathcal{O}(-q)_0).$$

Since \tilde{Y} is a quotient of a trivial pro-bundle, it suffices to take $q \geq 0$. We obtain the following, which, together with Theorem 1.4.11.(ii), immediately implies Theorem 1.4.17:

Proposition 2.0.21. For $q \geq 0$,

$$(2.0.22) \quad h(V_q; t) = \frac{t^{q(d-a)}(1 - t^{d-c})}{(1 - t^a)(1 - t^b)(1 - t^c)}.$$

Proof. We may identify V_q with the space of global sections of \tilde{Y} which vanish to order q at $(0, 1)$, using the injections of sheaves, $\mathcal{O}(n)_m \otimes \mathcal{O}(-1)_0 \cong \mathcal{O}(n-1)_m \hookrightarrow \mathcal{O}(n)_m$, which, on global sections, are the inclusions of sections vanishing at $(0, 1)$.³

To prove the proposition, we first rewrite the condition of vanishing to order q at $(0, 1)$, by explicitly describing the subspace $\{\alpha x + \beta y, \mathcal{O}_{\hat{X}}\}$. As in the introduction, let f_x, f_y, f_z denote the partial derivatives of $f \in \mathbb{C}[[x, y, z]]$ with respect to x, y , and z . We have $\{x, y\} = Q_z$, $\{y, z\} = Q_x$, and $\{z, x\} = Q_y$. Thus,

$$(2.0.23) \quad \{\alpha x + \beta y, f\} = \alpha(Q_z f_y - Q_y f_z) - \beta(Q_z f_x - Q_x f_z).$$

In particular, we deduce that

$$(2.0.24) \quad \{\alpha x + \beta y, \mathbb{C}[[x, y]]\} = \mathbb{C}[[x, y]]Q_z.$$

We may use this to compute the global sections of \tilde{Y} . Let T be a graded complement to $\mathbb{C}[[x, y]]$ in $\mathcal{O}_{\hat{X}}$, so that $\mathcal{O}_{\hat{X}} = \mathbb{C}[[x, y]] \oplus T$. By Lemma 2.0.19, the kernel of $\{\alpha x + \beta y, -\}$ lies in $\mathbb{C}[[x, y]]$. Hence, we have an exact sequence

$$(2.0.25) \quad 0 \rightarrow \mathcal{O}(-1)_a \otimes T \rightarrow \mathcal{O}_{\hat{X}}^0 / \mathbb{C}[[x, y]]Q_z \rightarrow Y \rightarrow 0,$$

³The map to sections vanishing at $(1, 0)$ comes from $\mathcal{O}(n-1)_{m+a-b} \hookrightarrow \mathcal{O}(n)_m$, so we could have instead used sections vanishing to order q at $(0, 1)$, but with weights shifted by $q(a-b)$. The argument goes through in the same way, swapping x with y and a with b .

and since Y is torsion-free, we conclude by taking global sections that

$$(2.0.26) \quad \Gamma(\mathbb{P}^1, Y) \cong \mathcal{O}_{\widehat{X}}^0 / \mathbb{C}[[x, y]]Q_z.$$

To describe V_q for $q > 0$, it will be convenient to sometimes work in the larger ring $\mathbb{C}[[x, y, z]][Q_z^{-1}]$, and define the operators

$$(2.0.27) \quad D_x := \partial_x - \frac{Q_x}{Q_z} \partial_z, \quad D_y := \partial_y - \frac{Q_y}{Q_z} \partial_z.$$

These operators make sense on $\mathbb{C}[[x, y, z]][Q_z^{-1}]/(Q)$ since $D_x = -\frac{1}{Q_z} \text{ad } y$ and $D_y = \frac{1}{Q_z} \text{ad } x$ (or because $D_x(Q) = 0 = D_y(Q)$). Moreover, it is clear that, if we think of z as implicitly dependent on x and y via $Q = 0$, then D_x and D_y are the derivatives with respect to x and y . In particular, on $\mathbb{C}[[x, y]]$, D_x and D_y restrict to the usual derivative with respect to x and y . Finally, we have

$$(2.0.28) \quad [D_x, D_y] = 0,$$

which follows from the above (one may also directly compute that $[D_x, D_y](z) = 0$).

It now follows from (2.0.23), (2.0.26), and (2.0.27) that V_q is identified with the solutions $G \in \mathcal{O}_{\widehat{X}}$ (modulo the subspace $\mathbb{C}[[x, y]]Q_z$) of the equations

$$(2.0.29) \quad \exists F_1, \dots, F_q \in \mathbb{C}[[x, y, z]]/(Q) \text{ s.t. } G = Q_z D_x F_1, \\ D_y F_1 = D_x F_2, \dots, D_y F_{q-1} = D_x F_q.$$

We break up most of the rest of the proof into lemmas. It suffices to consider *homogeneous* solutions to the above equations, which we do from now on. In particular, this means we can (and will) work in the uncompleted rings $\mathbb{C}[x, y, z]$, $\mathbb{C}[x, y, z]/(Q)$, $\mathbb{C}[x, y, z][Q_z^{-1}]/(Q)$, etc.

Lemma 2.0.30. Every homogeneous solution to (2.0.29) has the form

$$(2.0.31) \quad F_i = D_y^{i-1} D_x^{q-i} H, \quad G = Q_z D_x^q H,$$

for some homogeneous element $H \in \mathbb{C}[x, y, z]/(Q)$, which is uniquely determined by the F_i .

The proof mimics a standard proof of the Poincaré lemma.

Proof. We show, inductively on j , that there exist unique homogeneous $H_{i,j}$, for $i + j \leq q$ and $i \geq 1$, $j \geq 0$, such that (for $j \geq 1$)

$$(2.0.32) \quad D_x H_{i,j} = H_{i,j-1}, \quad D_y H_{i,j} = H_{i+1,j-1}, \quad H_{i,0} := F_i.$$

Then, it follows that $H = H_{1,q-1}$ has the desired property.

To do this, we use the formula, valid for all homogeneous f of positive degree:

$$(2.0.33) \quad |f|f = axD_x f + byD_y f.$$

Thus, given any g and h , there exists f such that $D_x f = g$ and $D_y f = h$ if and only if $D_y g = D_x h$, and in this case, $|f|f = axg + ayh$. The inductive step therefore follows by setting $g = H_{i,j}$ and $h = H_{i+1,j}$. \square

Next, we have to find what possible H can arise. This is answered by

Lemma 2.0.34. Let $H \in \mathbb{C}[x, y, z]$ be homogeneous. Then, the following are equivalent:

(i) For all polynomials f of degree $\leq n$, we have

$$(2.0.35) \quad f(D_x, D_y)H \in \mathbb{C}[x, y, z] + Q \cdot \mathbb{C}[x, y, z][Q_z^{-1}].$$

(ii) We have

$$(2.0.36) \quad H_z \in (Q^n) + (Q)_z.$$

Here and below, $(Q)_z$ is the partial derivative of the *ideal* $(Q) \subset \mathbb{C}[x, y, z]$, not the element.

As a consequence of the lemma, we deduce that the possible H in (2.0.31) are exactly those satisfying (2.0.36) for $n = q - 1$.

Proof. The implication (ii) \Rightarrow (i) is easy: since D_x, D_y are well-defined on $\mathbb{C}[[x, y, z]][Q_z^{-1}]/(Q)$, given (2.0.36), we may assume that $Q^n \mid H$, and (2.0.35) follows immediately.

Next, we prove (i) \Rightarrow (ii) inductively on n . For $n = 0$ the assertion is vacuous. Since the assertion does not depend on the choice of H modulo (Q) , we may assume inductively that $Q^{n-1} \mid H_z$ and (2.0.35) holds. We must prove that, up to adding an element of (Q) to H , we have $Q^n \mid H_z$.

Letting $f(D_x, D_y) = D_x$ in (2.0.35), we have

$$(2.0.37) \quad H_x - \frac{Q_x}{Q_z} H_z \in \mathbb{C}[x, y, z] + Q \cdot \mathbb{C}[x, y, z][Q_z^{-1}].$$

It follows that

$$(2.0.38) \quad Q_x H_z \in Q_z \mathbb{C}[x, y, z] + Q \cdot \mathbb{C}[x, y, z][Q_z^{-1}],$$

but since also $H_z \in \mathbb{C}[x, y, z]$, we in fact have

$$(2.0.39) \quad Q_x H_z \in (Q, Q_z).$$

Similarly, using $f(D_x, D_y) = D_y$ in (2.0.35), we have

$$(2.0.40) \quad Q_y H_z \in (Q, Q_z).$$

By the following Lemma 2.0.50, $H_z \in (Q, Q_z)$ itself. This proves (2.0.36) in the case $n = 1$.

We proceed now under the assumption that $n \geq 2$. Write $H_z = Q^{n-1}h$. Then

$$(2.0.41) \quad D_x H - H_x \in Q \cdot \mathbb{C}[x, y, z][Q_z^{-1}].$$

As a consequence, (2.0.35) implies that

$$(2.0.42) \quad f(D_x, D_y) H_x \in \mathbb{C}[x, y, z] + Q \mathbb{C}[x, y, z][Q_z^{-1}],$$

for all polynomials f of degree $\leq n - 1$. By the inductive hypothesis applied to H_x , we conclude that

$$(2.0.43) \quad H_{xz} \in (Q^{n-1}) + (Q)_z.$$

Substituting $H_z = Q^{n-1}h$, we find that

$$(2.0.44) \quad (n-1)Q^{n-2}Q_x h \in (Q^{n-1}) + (Q)_z.$$

Next, note that

$$(2.0.45) \quad (Q)_z \cap (Q^{n-2}) = (Q^{n-1})_z := \{g_z \mid g \in (Q^{n-1})\},$$

since if $Q \nmid g$, we have $\partial_z(Q^j g) = jQ^{j-1}Q_z g + Q^j g_z$, which is in (Q^{j-1}) but not (Q^j) . Applying this to (2.0.44), we deduce that

$$(2.0.46) \quad Q^{n-2}Q_x h \in (Q^{n-1}) + (Q^{n-1})_z = Q^{n-2}(Q, Q_z).$$

Dividing by Q^{n-2} , we get that

$$(2.0.47) \quad Q_x h \in (Q, Q_z),$$

and similarly,

$$(2.0.48) \quad Q_y h \in (Q, Q_z).$$

Thus, again applying Lemma 2.0.50 below, we find that $h \in (Q, Q_z)$, and hence

$$(2.0.49) \quad H_z \in Q^{n-1}(Q, Q_z) \subseteq (Q^n) + (Q)_z. \quad \square$$

The above proof rested on the following basic result:

Lemma 2.0.50. If $f \in \mathbb{C}[x, y, z]$ satisfies

$$(2.0.51) \quad Q_x f \in (Q, Q_z), \quad Q_y f \in (Q, Q_z),$$

then it follows that

$$(2.0.52) \quad f \in (Q, Q_z).$$

Proof. We claim that (2.0.51) implies that the ideal $(f) \subseteq \mathbb{C}[x, y, z]/(Q, Q_z)$ is a torsion module supported at the origin. Since (as we will recall), such torsion modules cannot be submodules of $\mathbb{C}[x, y, z]/(Q, Q_z)$, we will deduce that $f = 0$.

To prove the claim, note that, since the singularity at the origin is isolated, at every closed point in $Z(Q, Q_z)$ other than the origin, either Q_x or Q_y must be nonvanishing. Hence, in every local ring other than at the origin, either Q_x or Q_y is a unit. Thus, f is zero in every local ring other than the origin, i.e., (f) is a torsion module supported at the origin.

Next, note that, since Q is irreducible, $\mathbb{C}[x, y, z]/(Q)$ is a domain, and hence Q, Q_z form a regular sequence in $\mathbb{C}[x, y, z]$. That is, we have a Koszul resolution of $\mathbb{C}[x, y, z]/(Q, Q_z)$ of length two. Since any torsion module M supported at a point satisfies $\text{Ext}^i(M, \mathbb{C}[x, y, z]) = 0$ for $i < 3$, the long exact sequence of cohomology implies that $\text{Ext}^i(M, \mathbb{C}[x, y, z]/(Q, Q_z)) = 0$ for $i < 1$. Thus, $\text{Hom}((f), \mathbb{C}[x, y, z]/(Q, Q_z)) = 0$. This implies that $f = 0$, as desired. \square

To complete the proof of the proposition, first note that solutions to (2.0.29) such that $G \in \mathbb{C}[x, y]Q_z$, which form the subspace we wanted to quotient by, are exactly those for which the $F_i \in \mathbb{C}[x, y]$, and hence also $H \in \mathbb{C}[x, y]$. Therefore, to compute the Hilbert series of V_q , it remains, for each degree $|G| = m$, to find the dimension of the space of homogeneous H of degree $|G| + qa - (d - c)$ such that $Q^{q-1} \mid H_z$, modulo the space of such H which are polynomials in x and y . In other words, we seek the dimension of the space of elements H_z of degree $|G| + qa - d$ that are multiples of Q^{q-1} , modulo $(Q)_z$. By (2.0.45), this is equivalent to considering $(Q^{q-1}) = \{Q^{q-1}g \mid g \in \mathbb{C}[x, y, z]\}$ modulo $(Q^q)_z = \{Q^{q-1}(Qf_z + qQ_zf) \mid f \in \mathbb{C}[x, y, z]\}$. That is, writing $H_z = Q^{q-1}g$, our problem reduces to considering the space of polynomials g of degree $|g| = |G| + q(a - d)$ modulo $\langle Qf_z + qQ_zf \mid |f| = |g| + (c - d) \rangle$. We conclude that the Hilbert series of V_q is exactly (2.0.22). \square

3. PROOF OF THEOREM 1.1.13 WHEN X IS NOT OF TYPE A_{m-1}

We first consider the case where X is not a singularity of type A_{m-1} . The main step left is to give the promised proof of Lemma 1.4.3.

Proof of Lemma 1.4.3. Let $\mathcal{G}_X = \exp(\mathfrak{g}_X)$ be the group of Poisson automorphisms of $\mathcal{O}_{\hat{X}}$ generated by the flow of Hamiltonian vector fields ξ_f for $f \in \mathcal{O}_{\hat{X}}$ (so, $\xi_f|_g = \{f, g\}$, identifying the tangent space at every point of $\mathcal{O}_{\hat{X}}$ with $\mathcal{O}_{\hat{X}}$ itself). It is clear that $\mathcal{F}(X)^{\mathfrak{g}_X} = \mathcal{F}(X)^{\mathcal{G}_X} = \mathbb{C}[(V \setminus \{0\}) \times \mathcal{O}_{\hat{X}}^0]^{\mathcal{G}_X}$.

The idea behind the proof of the lemma is to view the fibers of Y as slices in $\alpha x + \beta y + \mathcal{O}_{\hat{X}}^0$ to the orbits of the group \mathcal{G}_X , in the following sense.

For a fixed (α, β) , note that $\alpha x + \beta y + \mathcal{O}_{\hat{X}}^0$ is stable under \mathcal{G}_X (since $\mathcal{O}_{\hat{X}}^0$ contains all Poisson brackets). Hence, we have a map $\mathcal{O}_{\hat{X}}/\mathcal{G}_X \rightarrow V$ with a canonical zero section $V := \langle x, y \rangle \subseteq \mathcal{O}_{\hat{X}}/\mathcal{G}_X$. Let U' be the pro-bundle over $V \setminus 0$ whose fiber at $\alpha x + \beta y$ is the tangent space to the fiber of the above map. In other words,

$$(3.0.53) \quad U'_{\alpha x + \beta y} = T_{\alpha x + \beta y}((\alpha x + \beta y + \mathcal{O}_{\hat{X}}^0)/\mathcal{G}_X).$$

The pro-bundle U' is evidently pulled back from a pro-bundle on \mathbb{P}^1 . Call this U .

Claim 3.0.54. (i) The punctured plane $V \setminus \{0\} \subseteq \mathcal{O}_{\hat{X}}/\mathcal{G}_X$ consists of smooth points.

(ii) We have a canonical isomorphism of pro-bundles

$$(3.0.55) \quad U \cong Y.$$

We will prove this claim below. For now, we assume it. Introduce the filtration on $\mathbb{C}[\mathcal{O}_{\hat{X}}/\mathcal{G}_X]$ by powers of the ideal I_V of functions vanishing on the plane V . By the claim, $V \setminus \{0\}$ consists of smooth points, and hence

$$(3.0.56) \quad \text{gr}_{I_V} \mathbb{C}[\mathcal{O}_{\hat{X}}/\mathcal{G}_X] = \mathbb{C}[U'] \cong \mathbb{C}[Y'],$$

where the latter denotes the global functions on the total space of the pro-bundle Y' . The total space of Y' (a pro-bundle over $V \setminus \{0\}$), is the same as the total space of $E = Y \oplus \mathcal{O}(-1)_a$ (a pro-bundle over \mathbb{P}^1), and we deduce that

$$(3.0.57) \quad \text{gr}_{I_V} \mathbb{C}[\mathcal{O}_{\hat{X}}/\mathcal{G}_X] \cong \mathbb{C}[E].$$

By Theorem 1.4.17, $\mathbb{C}[E]$ is in fact a polynomial algebra on homogeneous generators, finitely many in each degree. Hence, the lemma follows from

Claim 3.0.58. Let A be a graded commutative algebra A with a descending graded filtration $A = F_0 A \supseteq F_1 A \supseteq \cdots$ with $\cap_i F_i A = 0$ such that $\text{gr } A \cong \text{Sym } W$, where W is a bigraded vector space which is finite-dimensional in each bidegree. Then,

$$(3.0.59) \quad A = \text{Sym } \widetilde{W} \cong \text{gr } A,$$

for any graded lifting \widetilde{W} of W to A .

Proof. We have a canonical morphism of algebras, $\iota : \text{Sym } \widetilde{W} \rightarrow A$, which becomes an isomorphism when we take associated graded. Hence, it must be a monomorphism. To prove surjectivity, fix a degree $n \geq 0$. The surjectivity of $\text{Sym } W \rightarrow \text{gr } A$ says that

$$(3.0.60) \quad \iota(\text{Sym } \widetilde{W})_n + (F_{i+1} A)_n \supseteq (F_i A)_n, \forall i.$$

Since $\cap_i F_i A = 0$ and each $(F_i A)_n$ is finite-dimensional, there must exist $j \geq 0$ such that $(F_{j+1} A)_n = 0$. We deduce from (3.0.60) that $\iota(\text{Sym } \widetilde{W})_n \supseteq (F_j A)_n$, and applying (3.0.60) j more times, we deduce that $\iota(\text{Sym } \widetilde{W})_n \supseteq (F_0 A)_n = A_n$. \square

We apply this for $A = \mathbb{C}[\mathcal{O}_{\hat{X}}/\mathcal{G}_X]$, with the filtration by powers of I_V , and $A_0 = \mathbb{C}[E]$. \square

Proof of Claim 3.0.54. Since $\mathcal{O}_{\hat{X}}$ is a \mathcal{G}_X -representation, for any element $f \in \mathcal{O}_{\hat{X}}$, we have

$$(3.0.61) \quad T_{\alpha x + \beta y + \hbar f}(\mathcal{O}_{\hat{X}}/\mathcal{G}_X) = \{\alpha x + \beta y + \hbar f, \mathcal{O}_{\hat{X}}\}.$$

It suffices to show that the RHS is saturated as a $\mathbb{C}[[\hbar]]$ -module, i.e., if

$$(3.0.62) \quad \{\alpha x + \beta y + \hbar f, g\} \in \hbar \mathcal{O}_{\hat{X}}[[\hbar]],$$

then

$$(3.0.63) \quad g \in Z_{\alpha x + \beta y + \hbar f} + \hbar \mathcal{O}_{\hat{X}}[[\hbar]],$$

where $Z_{\alpha x + \beta y + \hbar f}$ is the Poisson centralizer. Since $Z_{\alpha x + \beta y} = \mathbb{C}[[\alpha x + \beta y]]$, (3.0.62) can only hold if $g \in \mathbb{C}[[\alpha x + \beta y]] + \hbar \mathcal{O}_{\hat{X}}[[\hbar]]$. Then, using that $\mathbb{C}[[\alpha x + \beta y + \hbar f]] \subseteq Z_{\alpha x + \beta y + \hbar f}$ (in fact, this is an equality), we see that (3.0.63) holds. \square

Now, Theorem 1.1.13 (in the non-type A case) follows from Theorem 1.4.17, since we have identified the regular invariant functions with the regular functions on the total space of $E = Y \oplus \mathcal{O}(-1)_a$. That is, we take the global sections of $\mathbf{Sym} E^* = \mathbf{Sym} Y^* \otimes \bigoplus_{m \geq 0} (\mathcal{O}(1)_{-a})^{\otimes m}$. If we decompose $Y = \prod_i \mathcal{O}(n_i)_{m_i}$, then the desired regular functions form a polynomial algebra on the generators $f_i \in \Gamma(\mathbb{P}^1, \mathcal{O}(-n_i)_{-m_i} \otimes (\mathcal{O}(1)_{-a})^{\otimes n_i}) \setminus \{0\}$, of weight $-m_i - n_i a$ and polynomial degree $n_i + 1$ (these vector spaces are one-dimensional, so any nonzero f_i will work), together with the generators α, β , which are the sections of $\mathcal{O}(1)_{-a}$. Letting L again denote the span of these generators, we see from (1.4.18) that

$$(3.0.64) \quad h(L; t^{-1}) = \sum_{i=0}^{\infty} s^{i+1} t^{ia} t^{i(d-a)} \frac{(1-t^{d-a})(1-t^{d-b})(1-t^{d-c})}{(1-t^a)(1-t^b)(1-t^c)} \\ = \frac{s(1-t^{d-a})(1-t^{d-b})(1-t^{d-c})}{(1-t^a)(1-t^b)(1-t^c)(1-t^d s)} = \frac{h(J_Q; t)s}{1-t^d s},$$

using the well-known formula for $h(J_Q; t)$ (which says that (Q_x, Q_y, Q_z) form a regular sequence in $\mathbb{C}[x, y, z]$). This completes the proof of Theorem 1.1.13 in the non- A_{m-1} case.

4. PROOF OF THEOREM 1.1.13 IN THE A_{m-1} CASE

This case involves the Poisson algebra $C[X] = \mathbb{C}[x, y, z]/(Q)$ for $Q = x^m + y^2 + z^2$. It will be convenient to present this slightly differently, as $\mathcal{O}_X = \mathbb{C}[x^m, xy, y^m] \subset \mathbb{C}[x, y]$, with the usual Poisson bracket $\{x, y\} = 1$ (this is the natural presentation from the point of view $X = \mathbb{C}^2/(\mathbb{Z}/m)$), and $|x| = |y| = 1$. As before, define $\mathcal{F}(X)$ as in (1.4.2). We once again have that $\mathbb{C}[\mathcal{O}_X]^{\mathfrak{g}_X}$ is (a completion of) $\mathcal{F}(X)^{\mathfrak{g}_X} \cong \mathcal{F}(X)^{\mathcal{G}_X}$, where $\mathcal{G}_X = \exp(\mathfrak{g}_X)$ is the group of Poisson automorphisms of $\mathcal{O}_{\hat{X}} = \mathbb{C}[[x^m, xy, y^m]]$ obtained from the flow of Hamiltonian vector fields. Also, since it will turn out that $\mathcal{F}(X)^{\mathfrak{g}_X}$ is finite-dimensional in each degree,⁴ in fact $\mathbb{C}[\mathcal{O}_X]^{\mathfrak{g}_X} \cong \mathcal{F}(X)^{\mathfrak{g}_X}$ as graded vector spaces. Now, Theorem 1.1.13 will follow from the following replacement for Theorem 1.4.17:

Proposition 4.0.65. The following set is a slice to the \mathcal{G}_X -orbits in $\mathbb{C}[[x^m, xy, y^m]]$ with nonzero coefficient of y^m or of xy :

$$(4.0.66) \quad y^m + \mathbb{C}[[x^m]] \langle 1, xy, (xy)^2, \dots, (xy)^{m-2} \rangle.$$

⁴This is also a consequence of the known fact that $HP_0(\mathbb{C}[\mathbb{C}^{2n}/(\mathbb{Z}/m)^n \rtimes S_n])$ is finite-dimensional, a special case of the result of the appendix to [BEG04], that $HP_0(\mathbb{C}[\mathbb{C}^{2n}]^G)$ is finite-dimensional for all finite $G < \mathbf{Sp}_{2n}$.

Proof. If the coefficient of y^m is zero but not the coefficient of xy , we can apply $e^{\xi(y^m)}$ to make the coefficient of y^m nonzero. Once we have a nonzero coefficient of y^m , by applying rescalings $y^m \mapsto \gamma y^m$, $x^m \mapsto \gamma^{-1} x^m$, we can make the coefficient of y^m one. Next, we use the lexicographical ordering \prec on monomials $\mathbb{C}[[x^m, xy, y^m]]$, where $x^a y^b \prec x^{a'} y^{b'}$ if either $a < a'$ or $a = a'$ and $b < b'$. Note that, for $a \neq 0$ and $f \in \mathbb{C}[[x^m, xy, y^m]]$ a power series with zero coefficient of y^m ,

$$(4.0.67) \quad \{x^a y^b, y^m + f\} = \{x^a y^b, y^m\} + \dots, \quad e^{\xi(x^a y^b)}(y^m + f) = \{x^a y^b, y^m\} + \dots,$$

where \dots denotes higher-order terms with respect to \prec . Hence, by applying elements $e^{\xi(x^a y^b)}$, we can kill off all monomials which appear in $\{y^m, \mathbb{C}[[x^m, xy, y^m]]\}$, which is a complement to $\mathbb{C}[[x^m]]\langle 1, xy, (xy)^2, \dots, (xy)^{m-2} \rangle$. Hence, any orbit with nonzero coefficient of either xy or y^m contains a point of the form (4.0.66).

It remains to prove that this point is unique. In other words, if $g \in \mathbb{C}[[x^m, xy, y^m]]$ satisfies

$$(4.0.68) \quad e^{\xi g}(y^m + f) = y^m + f', \quad f, f' \in \mathbb{C}[[x^m]]\langle 1, xy, (xy)^2, \dots, (xy)^{m-2} \rangle,$$

then $f = f'$. It suffices to show that, if $f \neq f'$, then the lowest-order term in $f - f'$ with respect to \prec lies in $\{y^m, \mathbb{C}[[x^m, xy, y^m]]\}$ (since this is impossible).

We assume that f and f' have no constant term. Note that, for any power series $R(y^m + f)$ in $y^m + f$, the operator $e^{\xi R(y^m + f)}$ fixes $y^m + f$. Using the Campbell-Baker-Hausdorff formula, we can replace g by an element g' such that $e^{\xi g'} = e^{\xi g} e^{\xi R(y^m + f)}$. Hence, inductively on \prec , we may assume that the coefficient in g of every monomial y^{km} is zero. In this case, the lowest-order term in $f - f'$ with respect to \prec appears in $\{y^m, g\}$. \square

Corollary 4.0.69. The invariant regular functions $\mathcal{F}(X)^{\mathcal{G}_X}$ restrict isomorphically to the regular functions on the slice (4.0.66).

Proof. By the proposition, restriction to the slice (4.0.66) identifies regular \mathcal{G}_X -invariant functions on the subvariety U of $\mathbb{C}[[x^m, xy, y^m]]$ consisting of power series whose coefficient of y^m or xy is nonzero with regular functions on (4.0.66). Since U is the complement of an affine subspace of codimension two, all regular (invariant) functions on U extend to regular (invariant) functions on all of $\mathbb{C}[[x^m, xy, y^m]]$. \square

It remains to compute the algebra of regular functions on (4.0.66). This is a polynomial algebra generated by the coordinate functions $w_{x^r y^s}$ of the slice, by which we mean that the point with coordinates $(w_{x^r y^s})$ in the slice is

$$(4.0.70) \quad y^m + \sum_{r,s} w_{x^r y^s} x^r y^s.$$

These do *not* necessarily have degree one as polynomial functions on $\mathcal{O}_{\hat{X}}$. To determine the degree we may consider the value of the slice coordinate $w_{x^r y^s}$ on γ times (4.0.70) for arbitrary $\gamma \in \mathbb{C}$. We can compute this by applying the element of \mathcal{G}_X which rescales by $y^m \mapsto \gamma^{-1} y^m$ and $x^m \mapsto \gamma x^m$. We deduce that the degree of $w_{x^r y^s}$ is $\frac{r-s}{m} + 1$. This yields exactly (1.1.14), proving Theorem 1.1.13 in the A_{m-1} case.

5. PROOF OF THEOREMS 1.2.1 AND 1.2.2 AND COROLLARY 1.2.3

To prove this, we first need to recall the structure of the zeroth Hochschild homology of symmetric products of algebras. Form the *coalgebra*

$$(5.0.71) \quad \mathcal{H}(A) := \bigoplus_{n \geq 0} \mathrm{HH}_0(\mathrm{Sym}^n A),$$

where the comultiplication map is given by the symmetrization maps

$$(5.0.72) \quad \mathrm{Sym}^p A \rightarrow \bigoplus_{m+n=p} \mathrm{Sym}^m A \otimes \mathrm{Sym}^n A.$$

(When $\mathrm{HH}_0(A)$ is finite-dimensional, this is dual to the symmetrization maps that we considered earlier.) We then have the following result:

Theorem 5.0.73. [EO06, Corollary 3.3] Let A be an infinite-dimensional simple algebra over \mathbb{C} with trivial center. Then, the algebra $\mathcal{H}(A)$ is a polynomial coalgebra,

$$(5.0.74) \quad \mathcal{H}(A) \cong \mathrm{Sym}(\mathrm{HH}_0(A)[t]),$$

where the isomorphism is the unique graded coalgebra map (with $\mathrm{HH}_0(A)$ and t both in degree one) such that the composition with the projection to $t^{n-1}\mathrm{HH}_0(A)$,

$$(5.0.75) \quad \mathrm{HH}_0(\mathrm{Sym}^n A) \rightarrow \mathrm{Sym}(\mathrm{HH}_0(A)[t]) \rightarrow t^{n-1}\mathrm{HH}_0(A),$$

has the form $[a_1 \& \cdots \& a_n] \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} t^{n-1} [a_{\sigma(1)} \cdots a_{\sigma(n)}]$.

We note that the above theorem is not stated in quite this way in [EO06], but rather in the equivalent formulation that

$$(5.0.76) \quad \mathrm{HH}_0(\mathrm{Sym}^n A) \cong \bigoplus_{\nu \in \mathcal{P}_n} \bigotimes_{i \geq 1} \mathrm{Sym}^{\nu_i} \mathrm{HH}_0(A),$$

where \mathcal{P}_n is the set of partitions of n , and ν_i denotes the number of cells of ν of size i .

We have the following immediate corollary:

Corollary 5.0.77. In the situation of Theorem 5.0.73, if $\mathrm{HH}_0(A)$ is finite-dimensional, then the commutative algebra $\bigoplus_{n \geq 0} \mathrm{HH}_0(\mathrm{Sym}^n A)^*$ is freely generated by the vector spaces $\langle [f^{\&n}] \mapsto T([f^n]) \rangle_{T \in \mathrm{HH}_0(A)^*} \subseteq \mathrm{HH}_0(\mathrm{Sym}^n A)^*$.

Proof of Theorem 1.2.1. The algebra Weyl_{2n}^H is well-known to be simple for all finite groups $H < \mathrm{Sp}_{2n}$. Indeed, since Weyl_{2n} is simple, so is the smash-product algebra $\mathrm{Weyl}_{2n} \rtimes H$, and this is therefore Morita equivalent to Weyl_{2n}^H . Obviously, Weyl_{2n}^H is also infinite-dimensional. Since, for finite $G < \mathrm{Sp}_2$ and $X = \mathbb{C}^2/G$, it is known that $\mathrm{HH}_0(\mathrm{Weyl}_2^G) \cong \mathrm{HP}_0(\mathcal{O}_X)$, (e.g. by comparing [AFLS00] for the former with the formulas mentioned in the introduction for the latter), the theorem follows from Theorem 5.0.73 and the main Theorem 1.1.13. \square

Proof of Theorem 1.2.2. Although the deformation quantization of \mathcal{O}_X is not, in general, simple, we may deform X to $Z(Q - \lambda)$, which is symplectic for $\lambda \neq 0$. Let $A_\lambda := \mathbb{C}[x, y, z]/(Q - \lambda)$, and let $A_{\lambda, \hbar}$ be its deformation quantization; the algebra $A_{\lambda, \hbar}[\hbar^{-1}]$ is simple. By results of Nest-Tsygan [NT95], we have $\mathrm{HH}_0(A_{\lambda, \hbar}[\hbar^{-1}]) \cong \mathrm{HP}_0(A_\lambda)((\hbar))$ for $\lambda \neq 0$, i.e., the Brylinski spectral sequence degenerates. Moreover, the Betti numbers of A_λ are 1, 0, and μ_Q , where $\mu_Q = \dim \mathbb{C}[x, y, z]/(Q_x, Q_y, Q_z)$ is the Milnor number of X . Hence, $\dim_{\mathbb{C}((\hbar))} \mathrm{HH}_0(A_{\lambda, \hbar}[\hbar^{-1}]) = \dim \mathrm{HP}_0(A_0)$. By Theorem 5.0.73 and Theorem 1.1.13,

we deduce that $\mathrm{HH}_0(\mathrm{Sym} A_{\lambda, \hbar}[\hbar^{-1}]) \cong \mathrm{HP}_0(\mathrm{Sym} A_0)((\hbar))$ as graded algebras (with degree n corresponding to Sym^n , so not looking at the grading on A_0 yet). However, as $\lambda \rightarrow 0$, $\mathrm{HH}_0(\mathrm{Sym}^n A_{\lambda, \hbar}[\hbar^{-1}])$ can only increase in dimension over $\mathbb{C}((\hbar))$, but the Brylinski spectral sequence shows that $\dim_{\mathbb{C}((\hbar))} \mathrm{HH}_0(\mathrm{Sym}^n A_{0, \hbar}[\hbar^{-1}]) \leq \dim \mathrm{HP}_0(\mathrm{Sym}^n A_0)$ for all n . Hence, the dimensions are equal, and the Brylinski spectral sequence degenerates. \square

Proof of Corollary 1.2.3. In a formal punctured neighborhood of $\gamma = 0$, we see from the above that the zeroth Hochschild homology is constant and as above. Hence, by a standard argument, the same is true when \hbar is replaced by actual values of γ that do not obey a countable number of polynomial equations, i.e., for all but countably many γ , and the zeroth Hochschild homology is as above. \square

Remark 5.0.78. We deduce from the above that, for the elliptic algebras A_γ , the two sides of (5.0.74) are abstractly isomorphic as bigraded algebras (Corollary 1.2.3). However, the map defined in Theorem 5.0.73 is *not* an isomorphism: for example, consider $T \in \mathrm{HH}_0(A)^*$ which takes the degree-zero coefficient of an element of $A/[A, A]$. Then, $T([a^2]) = T([a])^2$, so the map is not an isomorphism (cf. Corollary 5.0.77). Moreover, the map of Theorem 5.0.73 is not even a bigraded map: while it preserves degree (and is thus a graded map), it does not preserve weight: while $|t^{n-1}T| = -(n-1)d + |T|$ on the RHS for $T \in \mathrm{HH}_0(A_\gamma)^*$, the element $[f^{\&n}] \mapsto T([f^n])$ has degree $|T|$ on the LHS. So the fact that an isomorphism between the two sides of (5.0.74) exists is subtle and there may not be a canonical one.

6. ACKNOWLEDGEMENTS

We thank Qingchun Ren for writing computer programs which allowed us to compute HP_0 for certain surfaces in low degrees (as part of an MIT UROP with the authors). We are grateful to J. Alev for useful discussions. The first author is partially supported by NSF grant DMS-0504847. The second author is a five-year fellow of the American Institute of Mathematics.

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